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Solvable quantum mechanical model in two-dimensional space

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Abstract

A one-particle non-relativistic quantum mechanical solvable model in two-dimensional space is given. The Hamiltonian is the sum of kinetic and interaction parts. Interactions are separable and can be centred at n arbitrary points of the plane. Conditions for the existence and for the number of bound states in finite linear chains are formulated in terms of the parameters of the interactions and intercentre distances. Scattering problems are also considered. Finally, when the interactions are centred in a single centre, it is shown that the model remains solvable in the presence of a uniform magnetic field of arbitrary intensity.

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1. Introduction

In the few existing solvable quantum mechanical models for one particle, Hamiltonians based on the zero-range interactions, also called point interactions or Fermi pseudo-potentials, have been proved to be fruitful for both mathematical developments and physical applications [1]. Another solvable model in three-dimensional space, with non-zero-range separable interactions centred at n arbitrary points has been proposed [2], developed and applied [3–7]. It is the purpose of this paper to show that an analogue model can be formulated and solved in two-dimensional space. On the one hand, the solution is simpler in two dimensions because the complexity due to the algebra of angular momentum disappears. On the other hand, the solution is more difficult in two dimensions because the matrix elements of the free resolvent involve irregular Bessel functions which have branch cut on the negative real axis, whereas the spherical Bessel functions occurring in the three-dimensional case do not have such branch cut. The present model is solvable in the sense that scattering and bound state problems reduce to the evaluation, or to the numerical search of zeros, or to the diagonalization of finite matrices whose elements are known analytic functions.

In addition to the intrinsic interest related to the change of space dimensions, the formulation of the model in two-dimensional space has a specific interest because it should allow us to take into account the effect of an exterior uniform magnetic field more easily than in the three-dimensional case. Magnetic field effects in the framework of zero-range interaction have been considered recently [8] (see also [9–11]). The main purpose of this paper is to study the field free case ($B = 0$). In the last section however, it is shown that in the case of interactions centred at one point only, the present model is solvable for the arbitrary value of the uniform magnetic field. The spectrum from the low field limit (linear Zeeman limit) to the infinite field limit (Landau levels limit) can be determined according to a unique procedure based on the explicit expression of a partial wave Green function for a charged particle in a uniform magnetic field in two-dimensional space. Finally, it is needless to say that models in two-dimensional space are relevant to surface physics.

2. The Hamiltonian

We shall mainly use notations similar to those used for the three-dimensional case [2]. The two Cartesian coordinates will be denoted by x, y, related to the polar coordinates r, φ by $x = r\cos(\varphi)$, $y = r\sin(\varphi)$, $0 \leqslant r < \infty$ and $0 \leqslant \varphi < 2\pi$. The Hamiltonian then reads

$$H = \frac{p_x^2 + p_y^2}{2M} + \sum_j^{N_c} V_j = \frac{p^2}{2M} + V.$$

Its kinetic part $p^2/2M$ will be referred to as H_0, the remaining part as V. The interaction V_j is centred at the point P_j with $\mathbf{a}_j \equiv \overrightarrow{OP_j}$. A one-to-one correspondence between the N_c points P_j and indices j is assumed. Atomic units are used ($\hbar = 1, e = 1, M_{e^-} = 1$). The interaction V_j is a sum of separable interactions or projectors defined by

$$V_j = \sum_k \lambda_j^k \left|\xi_j^k\right\rangle\left\langle\xi_j^k\right|$$
$$\left|\xi_j^k\right\rangle = \exp(-\mathrm{i}\mathbf{a}_j \cdot \mathbf{p}) r_j^k \left|r_j^k, m_j^k\right\rangle.$$

The kets $\left|r_j^k, m_j^k\right\rangle$ are generalized eigenvectors of the radial position operator r, with eigenvalue r_j^k, and the eigenvector of the angular momentum operator

$$\ell = xp_y - yp_x, \tag{1}$$

with eigenvalue m_j^k. The expression generalized vector means a normalization with a Dirac distribution. The operator $\exp(-\mathrm{i}\mathbf{a}_j \cdot \mathbf{p})$ is the operator which translates the state by the displacement $\mathbf{a}_j$.

The total number of projectors $\left(\sum_{j,k} 1\right)$ in the Hamiltonian will be denoted by N_p. For convenient notations, the Greek indices α, β, γ will correspond to a couple j, k. For example $\lambda_j^k \left|\xi_j^k\right\rangle\left\langle\xi_j^k\right| = \lambda_\alpha |\xi_\alpha\rangle\langle\xi_\alpha|$, $r_j^k = r_\alpha$, $m_j^k = m_\alpha$. Thus α can vary between 1 and N_p.

If T denotes the time reversal operator, the relation

$$T|r_\alpha, m_\alpha\rangle = (-1)^{m_\alpha}|r_\alpha, -m_\alpha\rangle \tag{2}$$

implies that for a Hamiltonian invariant under time reversal, each V_j must involve $|r_\alpha, m_\alpha\rangle\langle r_\alpha, m_\alpha| + |r_\alpha, -m_\alpha\rangle\langle r_\alpha, -m_\alpha|$.

Each projector is invariant with respect to rotations *with respect to its centre* P_j since

$$\exp(-\mathrm{i}\varphi\ell)|r, m\rangle = \exp(-\mathrm{i}\varphi m)|r, m\rangle.$$

Besides the kets already introduced, we shall need generalized eigenvectors of the position operator $\mathbf{r}$, of the momentum operator $\mathbf{p}$ and of the radial momentum operator p. These vectors

and their normalization and the relation between them are now summarized. For the sake of convenience, the indices j, k have been suppressed:

$$\langle r'm'|r,m\rangle = \delta_{m'm}\frac{\delta(r'-r)}{r} \tag{3}$$

$$\langle \mathbf{r}'|\mathbf{r}\rangle = \delta(\mathbf{r}'-\mathbf{r}) = \delta(x'-x)\delta(y'-y) \tag{4}$$

$$\langle \mathbf{r}'|r,m\rangle = \frac{\delta(r'-r)}{r}\frac{\exp(\mathrm{i}m\varphi')}{\sqrt{2\pi}} \tag{5}$$

$$\mathrm{I} = \int_0^\infty \mathrm{d}r\, r \sum_{m=-\infty}^{\infty} |r,m\rangle\langle r,m| = \int \mathrm{d}^2 r\, |\mathbf{r}\rangle\langle\mathbf{r}| \tag{6}$$

$$|r,m\rangle = \int_0^{2\pi} \mathrm{d}\varphi\, \frac{\exp(\mathrm{i}m\varphi)}{\sqrt{2\pi}}|\mathbf{r}\rangle \tag{7}$$

$$|\mathbf{r}\rangle = \sum_{m=-\infty}^{\infty} \frac{\exp(-\mathrm{i}m\varphi)}{\sqrt{2\pi}}|r,m\rangle \tag{8}$$

and the same equations with p in place of r, for example $\langle p'm'|p,m\rangle = \delta_{m'm}\frac{\delta(p'-p)}{p}$. Equations (7) and (8) are direct consequences of equations (5) and (6). It remains to give the relations between the bases involving the momentum and the bases involving the position:

$$\langle \mathbf{r}|\mathbf{p}\rangle = \frac{1}{2\pi}\exp(\mathrm{i}\mathbf{p}\cdot\mathbf{r}) \tag{9}$$

$$\langle \mathbf{p}|r,m\rangle = (-\mathrm{i})^m J_m(pr)\frac{\exp(\mathrm{i}m\varphi_{\mathbf{p}})}{\sqrt{2\pi}} \tag{10}$$

$$\langle \mathbf{r}|p,m\rangle = \mathrm{i}^m J_m(pr)\frac{\exp(\mathrm{i}m\varphi_{\mathbf{r}})}{\sqrt{2\pi}} \tag{11}$$

$$\langle rm|p,m'\rangle = \delta_{m'm}\mathrm{i}^m J_m(pr). \tag{12}$$

J_m is the Bessel function regular at the origin as defined in [12]. The constants of normalization and phase factors on the right-hand sides of equations (10)–(12) are completely determined by equations (3)–(9) due to the following series [13]:

$$\exp(\mathrm{i}z\cos(\varphi)) = \sum_{n=-\infty}^{\infty} \mathrm{i}^n \exp(\mathrm{i}n\varphi) J_n(z). \tag{13}$$

For example, equation (10) is obtained as follows:

$$\begin{aligned}\langle \mathbf{p}|r,m\rangle &= \int_0^{2\pi} \mathrm{d}\varphi_{\mathbf{r}}\frac{\exp(\mathrm{i}m\varphi_{\mathbf{r}})}{\sqrt{2\pi}}\langle\mathbf{p}|\mathbf{r}\rangle \\ &= \int_0^{2\pi} \mathrm{d}\varphi_{\mathbf{r}}\frac{\exp(\mathrm{i}m\varphi_{\mathbf{r}})}{2\pi\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}(-\mathrm{i})^n J_n(pr)\exp(-\mathrm{i}n(\varphi_{\mathbf{r}}-\varphi_{\mathbf{p}})) \\ &= (-\mathrm{i})^m J_m(pr)\frac{\exp(\mathrm{i}m\varphi_{\mathbf{p}})}{\sqrt{2\pi}}.\end{aligned}$$

3. Solution in terms of matrix elements of the resolvent

It is known (see e.g. [2]) how the bound states problems and scattering problems for a Hamiltonian which is the sum of H_0 and separable interactions reduce to the knowledge of the matrix elements of the free resolvent,

$$G_0(z) = \frac{1}{z - H_0}.$$

Specifically, we need the following matrix elements:

$$\langle r, m|G_0(z)|r', m'\rangle = \delta_{mm'} \int_0^\infty \mathrm{d}p\, p \frac{J_m(pr)J_m(pr')}{z - \frac{p^2}{2M}} \tag{14}$$

$$\langle r, m|\exp(\mathrm{i}\mathbf{a}\cdot\mathbf{p})G_0(z)|r', m'\rangle = -(-1)^{m-m'} 2M \exp(\mathrm{i}\varphi_{\mathbf{a}}(m' - m)) \int_0^\infty \mathrm{d}p\, p \frac{J_m(pr)J_{m'}(pr')J_{m-m'}(ap)}{p^2 - 2Mz}, \tag{15}$$

with $\mathbf{a}$ the vector joining the centre P' to the centre P. Equation (14) is obtained by inserting a closure relation (see equation (6) with p in place of r) and from equation (12). Equation (15) is obtained from

$$\exp\{-\mathrm{i}a[\cos(\varphi_{\mathbf{a}})p_x + \sin(\varphi_{\mathbf{a}})p_y]\} = \exp(-\mathrm{i}\varphi_{\mathbf{a}}\ell_z)\exp(-\mathrm{i}ap_x)\exp(\mathrm{i}\varphi_{\mathbf{a}}\ell_z),$$

which itself results from the commutation relations

$$[\ell_z, p_x] = \mathrm{i}p_y \qquad [\ell_z, p_y] = -\mathrm{i}p_x$$

and from equation (13). The final expression for $\langle r, m|G_0(z)|r', m'\rangle$ is well known and can be obtained either by explicit evaluation of the integral (for example by using the result of appendix A) or by solving the differential equation for this partial wave Green function and by taking into account appropriate boundary conditions. This last method is described elsewhere, see e.g. [14, 15], and will be used in the last section of this paper for the computation of matrix elements of the resolvent $G_{0B}(z)$ in the presence of a magnetic field $\mathbf{B}$. For the matrix element between different centres, the explicit evaluation of the integral by the result of the appendix can be done in the case $a \geqslant r + r'$, which is always supposed in this paper. This inequality corresponds to non-overlapping interactions. From now on, we use the notation p_z for the following square root with the positive imaginary part:

$$p_z \equiv \sqrt{2Mz}.$$

When z is negative, p_z is thus on the positive imaginary axis. When z is real positive, the limit $\lim_{\epsilon\to 0} z + \mathrm{i}\epsilon$ has to be taken in the evaluation of integral, where ϵ is positive. The final results then are

$$\langle r, m|G_0(z)|r', m'\rangle = \delta_{mm'}[-\mathrm{i}M\pi J_m(p_z r_<)H_m^{(1)}(p_z r_>)] \tag{16}$$

$$\langle r, m|\exp(\mathrm{i}\mathbf{a}\cdot\mathbf{p})G_0(z)|r', m'\rangle = \mathrm{i}(-1)^{m-m'+1} M\pi \exp(\mathrm{i}\varphi_{\mathbf{a}}(m' - m)) H_{m-m'}^{(1)}(ap_z)J_m(rp_z)J_{m'}(r'p_z). \tag{17}$$

$r_<$ and $r_>$ respectively denotes the smallest and greatest value among r, r', or their common value if they are equal. It is stressed that equation (17) is the essential result which allows the present model to be solvable for the multicentre case. This equation has been obtained as a particular case of the general property derived in the appendix. $H_m^{(1)}$ is the usual notation [12] for a solution of the Bessel differential equation, called the Hankel function, irregular at the

origin, whose asymptotic behaviour is given in the appendix, equation (A.2). For scattering problems, it will be useful to recall that $H_m^{(1)}(z) = J_m(z) + \mathrm{i}Y_m(z)$ with both $J_m(z)$ and $Y_m(z)$ real for the real positive argument z.

For normalization of bound states, one also needs the matrix elements $\langle\xi_j|G_0^2(z)|\xi_k\rangle$, and for orthonormalization between different states or consideration of time evolution of wave packets [7], one also needs the matrix elements $\langle\xi_j|G_0(z_1)G_0(z_2)|\xi_k\rangle$. These matrix elements can be obtained most simply from the algebraic and analytic identities:

$$G_0(z_1)G_0(z_2) = G_0(z_2)G_0(z_1) = -\frac{G_0(z_2) - G_0(z_1)}{z_2 - z_1}$$

$$(G_0(z))^{n+1} = -\frac{1}{n}\frac{\partial(G_0(z))^n}{\partial z}.$$

The derivatives of Bessel and Hankel functions satisfy [12] the relation

$$\mathcal{C}_{m-1}(x) - \mathcal{C}_{m+1}(x) = 2\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{C}_m(x),$$

with $\mathcal{C}_m$ for J_m or $H_m^{(1)}$. The matrix elements $\langle\xi_j|G_0^2(z)|\xi_k\rangle$ can therefore also be expressed in terms of Bessel and Hankel functions.

4. Bound states problems

The bound state energies for a finite number of centres, or equivalently the poles on the negative real axis of the resolvent

$$G(z) = [z - p^2/(2M) - V]^{-1},$$

can be determined [6] as the negative z values for which the determinant of a matrix $b(z)$ of order equal to the number N_b of projectors are zero. This matrix $b(z)$ is defined by its matrix elements:

$$b_{\alpha,\beta}(z) = \delta_{\alpha\beta} - \lambda_\beta\langle\xi_\alpha|G_0(z)|\xi_\beta\rangle. \tag{18}$$

These results follow from the identity $G(z) = G_0(z) + G_0(z)VG(z)$. The exact normalized eigenvector of H corresponding to the eigenvalue z_u is [7]

$$|\psi_u\rangle = \frac{G_0(z_u)\sum_\alpha|\xi_\alpha\rangle\langle\xi_\alpha|\psi_u\rangle}{\sqrt{\sum_{\alpha,\beta}\langle\psi_u|\xi_\alpha\rangle\langle\xi_\alpha|G_0^2(z_u)|\xi_\beta\rangle\langle\xi_\beta|\psi_u\rangle}}. \tag{19}$$

The coefficients $\langle\xi_\alpha|\psi_u\rangle$ appearing in the numerator and in the denominator are the elements of a column eigenvector associated with the zero eigenvalue of the matrix $b(z_u)$. For linear chains on the x axis, φ_a is zero, and for $z < 0$ this matrix then is real, so that $\langle\xi_\alpha|\psi_u\rangle$ can all be chosen real, this choice being made from now on.

The wavefunction $\langle\mathbf{r}|\psi_u\rangle$ is obtained from equation (19) and

$$\begin{aligned}\langle\mathbf{r}|G_0(z_u)|\xi_\alpha\rangle &= \langle\mathbf{r}|\exp(-\mathrm{i}\mathbf{a}_j\cdot\mathbf{p})G_0(z_u)r_j^k\big|r_j^k, m_j^k\big\rangle \\ &= r_j^k\frac{\exp\left(\mathrm{i}m\varphi_{\mathbf{r}-\mathbf{a}_j}\right)}{\sqrt{2\pi}}\big\langle|\mathbf{r}-\mathbf{a}_j|, m_j^k|G_0(z)|r_j^k, m_j^k\big\rangle.\end{aligned}$$

Let us also recall general results that have been obtained in [6] for the Hamiltonian ${}_{N_p}H = H_0 + \lambda\sum_{\alpha=1}^{N_b}|\xi_\alpha\rangle\langle\xi_\alpha|$, which are independent of the dimension of space.

- The energy of a bound state is an increasing function of λ.
- There are at most N_p bound states.

- The critical values ${}_{N_p}\lambda_k$ for which a zero energy state exists for a Hamiltonian with N_p projectors interlace the critical values ${}_{N-1}\lambda_k$ for a Hamiltonian with $N-1$ projectors.

For the study of conditions for bound states, the small z behaviour of the matrix $b(z)$ will be needed. In this limit [12, 13],

$$J_m(z) \sim s(m)\frac{\left(\frac{z}{2}\right)^{|m|}}{\Gamma(|m|+1)} \tag{20}$$

$$H^{(1)}_{m\neq 0}(z) \sim s(m)\left(\frac{-\mathrm{i}}{\pi}\right)\Gamma(|m|)\left(\frac{z}{2}\right)^{-|m|} \tag{21}$$

$$H^{(1)}_{0}(z) \sim \mathrm{i}\frac{2}{\pi}\log(z), \tag{22}$$

where $f(z) \sim g(z)$ means that $f(z)/g(z)$ tends to unity in the limit considered, and where the function s is defined by

$$s(m) = (-1)^{\frac{m-|m|}{2}}. \tag{23}$$

The small z behaviour of the diagonal elements are thus

$$b_{\alpha\alpha}(0) = 1 + \frac{M\lambda_\alpha r_\alpha^2}{|m_\alpha|} \qquad \text{if} \quad m_\alpha \neq 0 \tag{24}$$

$$b_{\alpha,\alpha}(z) \sim 1 - 2M\lambda_\alpha r_\alpha^2 \log(p_z r_\alpha) \qquad \text{if} \quad m_\alpha = 0. \tag{25}$$

For the small z behaviour of non-diagonal elements two cases have to be considered.

First, $\alpha = j, k \neq \beta = j, \ell$ (same centre but different projectors). Then for $m_\alpha \neq 0$

$$b_{\alpha,\beta}(z) \sim \delta_{m_\alpha m_\beta}\frac{M\lambda_\beta r_\alpha r_\beta}{|m_\alpha|}\left(\frac{r_<}{r_>}\right)^{|m|} \tag{26}$$

and for $m_\alpha = 0$

$$b_{\alpha,\beta}(z) \sim -\delta_{m_\alpha m_\beta} 2M\lambda_\beta r_\alpha r_\beta \log(p_z r_>) \sim -\delta_{m_\alpha m_\beta} 2M\lambda_\beta r_\alpha r_\beta \log(p_z). \tag{27}$$

Second, $\alpha = j, k$, $\beta = i, \ell$, $i \neq j$ (different centres). Then for $m_\alpha = m_\beta$

$$b_{\alpha,\beta}(z) \sim -\frac{2^{1-2|m_\alpha|}M\lambda_\beta r_\alpha^{1+|m_\alpha|} r_\beta^{1+|m_\alpha|}}{(|m_\alpha|!)^2} p_z^{2|m_\alpha|}\log(|\mathbf{a}_j - \mathbf{a}_i| p_z) \tag{28}$$

and for $m_\alpha \neq m_\beta$

$$\begin{aligned} b_{\alpha,\beta}(z) \sim{}& (-1)^{m_\alpha - m_\beta} s(m_\alpha - m_\beta) s(m_\alpha) s(m_\beta) \exp\left(\mathrm{i}\varphi_{\mathbf{a}_j - \mathbf{a}_i}(m_\beta - m_\alpha)\right) \\ & 2^{|m_\alpha - m_\beta| - |m_\alpha| - |m_\beta|} M\lambda_\beta \frac{r_\alpha^{1+|m_\alpha|} r_\beta^{1+|m_\beta|}}{(|\mathbf{a}_j - \mathbf{a}_i|)^{|m_\alpha - m_\beta|}} \\ & \frac{(|m_\alpha - m_\beta| - 1)!}{|m_\alpha|!|m_\beta|!}(p_z)^{|m_\alpha|+|m_\beta|-|m_\alpha - m_\beta|}. \end{aligned} \tag{29}$$

Let us now apply these general results for bound states to different cases.

4.1. Conditions for the bound state for a single projector

The Hamiltonian $H_0 + \lambda|\xi\rangle\langle\xi|$ can support at most one bound state and supports it if

$$0 > \frac{1}{\lambda} > \langle\xi|G_0(0)|\xi\rangle.$$

The relations [13]

$$J_{-m}(z) = (-1)^m J_m(z) \qquad H^{(1)}_{-m}(z) = (-1)^m H^{(1)}_m(z)$$

clearly verify that the energy does not depend on the sign of m. This result can be obtained more generally from equation (2).

Equation (16) and, for $|m| > 0$, equations (20)–(21) prove that $H_0 + \lambda r^2 |r, m\rangle\langle r, m|$ is able to support a bound state if

$$\lambda < \lambda_c = -\frac{|m|}{Mr^2}. \tag{30}$$

For $m = 0$, equation (22) must be used in place of equation (21) and since then $\lim_{z\to 0} z^2 J_0(z) H^{(1)}_0(z) = 0$, equation (30) remains true for $m = 0$. The fact that for $m = 0$, a bound state can exist for $\lambda < 0$ arbitrary close to 0 should be contrasted with the three-dimensional case where the critical value for s partial wave ($\ell = 0$) is [2] strictly negative, $-1/(2Mr^2)$.

At this stage, it is of interest to compare the energy of the bound state in two and three dimensions for the same values of the parameters. It has been shown in [3] that the band structure of lithium can be reasonably reproduced with $\lambda = -0.398\,909$ and $r = 1.817\,6943$ and an appropriate choice for the lattice constant. It has also been found [3] that for these particular values of the parameters λ and r the bound state energy for a single centre s state is $-0.217\,17$. Using the same values of these parameters λ and r for comparison, the present two-dimensional model gives a bound state energy equal to $-0.295\,308$ for $m = 0$.

For future illustration of the magnetic field effect in the last section, we shall consider the λ values which for the mass parameter $M = 1$, and the range parameter $r = 1$, give a bound state energy equal to -1 for $|m| = 0, 1, \ldots, 5$. These values are obtained by solving the equation $\frac{1}{\lambda} = \langle r, m|G_0(-1)|r, m\rangle$. They are

$\lvert m\rvert$	0	1	2	3	4	5
$\lambda^{\lvert m\rvert}$	$-1.335\,05$	$-1.769\,66$	$-2.485\,21$	$-3.337\,05$	-4.2539	$-5.202\,68.$

(31)

4.2. Bound states in a finite linear chain

4.2.1. The case $m = 0$ only. The Hamiltonian with K centres and K projectors can be written as

$${}_K H^0 = \frac{p^2}{2M} + \lambda \sum_{j=1}^{K} |\xi^0_j\rangle\langle\xi^0_j| \qquad |\xi^0_j\rangle = \exp(-\mathrm{i} j L p_x) r |r, 0\rangle.$$

There is in this case a one-to-one correspondence between centres and projectors and the Greek indices can be replaced by Latin indices. In the zero energy limit, the diagonal terms of the matrix b are given by equation (25). The behaviour of the non-diagonal terms is obtained from equation (27):

$$b_{ij}(z) = -2M\lambda r^2 \log(|j - i| L p_z).$$

Both diagonal and non-diagonal terms have logarithmic divergence as z goes to zero. After division of each matrix element by $2Mr^2\lambda \log(p_z)$ and with $x \equiv 1/\log(p_z)$, the condition for the bound state in the limit of zero energy reads $\lim_{x\to 0} \det(f(x)) = 0$ with the matrix elements of the symmetric matrix $f(x)$ equal to

$$(f(x))_{ii} = -1 + x\left(\frac{1}{2Mr^2\lambda} - \log(r)\right)$$
$$(f(x))_{ij} = -1 - x\log(|\beta - \alpha| L) = -1 - x\log(L) - x\log(|j - i|),$$

with $|j-i| = 1, 2, \ldots$ the absolute value of the difference between the indices of two different centres. The condition $\lim_{x\to 0} \det(f(x)) = 0$ is thus equivalent to the requirements that, in the limit where x goes to zero, $1 - x\left(\frac{1}{2Mr^2\lambda} - \log(r)\right)$ be the eigenvalue of the matrix $g(x)$ with zero diagonal elements and non-diagonal elements equal to those of the matrix $f(x)$. Let $\gamma_k(x)$ be the eigenvalues of $g(x)$, so that the critical values λ_k where a bound state appears or disappears are now given by

$$\frac{1}{\lambda_k} = 2Mr^2 \lim_{x\to 0}\left(\frac{1-\gamma_k(x)}{x} + \log(r)\right). \tag{32}$$

The matrix $g(x)$ can be decomposed as

$$g(x) = g_0(x) - xh,$$

where $g_0(x)$ is a matrix with diagonal terms equal to zero and $-1 - x\log(L)$ for all the non-diagonal terms, and where h a matrix with diagonal terms equal to zero and

$$h_{ij} = \log(|j-i|) \qquad \text{for} \quad i \neq j. \tag{33}$$

The matrix $g_0(x)$ has two eigenvalues, $(K-1)(-1-x\log(L))$, which is non-degenerated, and $1 + x\log(L)$, which has a degree of degeneracy equal to $K-1$. In the limit where x goes to zero, $(1-\gamma_k(x))/x$ on the right-hand side of equation (32) thus diverges for the non-degenerated eigenvalue, and therefore $\lambda = 0$ is the critical value where the first bound state appears or disappears. Since we are concerned with the limit where x goes to zero, the exact results for the other eigenvalues of $g(x)$ can be obtained from the first-order degenerate perturbation theory. The matrix $\Lambda^{-1}g_0(x)\Lambda$ is diagonal with the first diagonal term equal to the non-degenerated eigenvalue, the other diagonal terms equal to the degenerated eigenvalue, for the matrix Λ defined as follows. The matrix elements of the first column of Λ are equal to unity and, for example, the matrix elements of the first line of Λ are equal to unity, the diagonal matrix elements except the first one are equal to minus unity, all the other matrix elements being equal to zero,

$$\Lambda_{i1} = 1 = \Lambda_{1i} \qquad \Lambda_{i>1,j>1} = -\delta_{ij}.$$

The inverse Λ^{-1} of Λ can be evaluated,

$$(\Lambda^{-1})_{i\neq j} = \frac{1}{K} = (\Lambda^{-1})_{11} \qquad (\Lambda^{-1})_{i>1,i} = -1 + \frac{1}{K}.$$

Degenerated perturbation theory tells us that the perturbation has to be diagonalized in the subspace spanned by the eigenvectors of the unperturbed operator. We therefore introduce the submatrix U of order $K-1$ of $\Lambda^{-1}h\Lambda$ whose matrix elements are defined by

$$U_{ij} = (\Lambda^{-1}h\Lambda)_{i-1,j-1}.$$

The non-zero critical value of λ is thus finally given by

$$\frac{1}{\lambda_k} = 2Mr^2\left(\log\left(\frac{r}{L}\right) + u_k\right), \tag{34}$$

with u_k the eigenvalues of the matrix U of order $K-1$. We stress that these critical values λ_k are exact, although we use the first-order perturbation theory, because we are concerned with the limit where x goes to zero.

For example, for $M = r = 1, L = 3, m = 0$ and a linear chain with four centres the critical values are $0, -0.197\,446, -0.402\,43, -0.654\,876$.

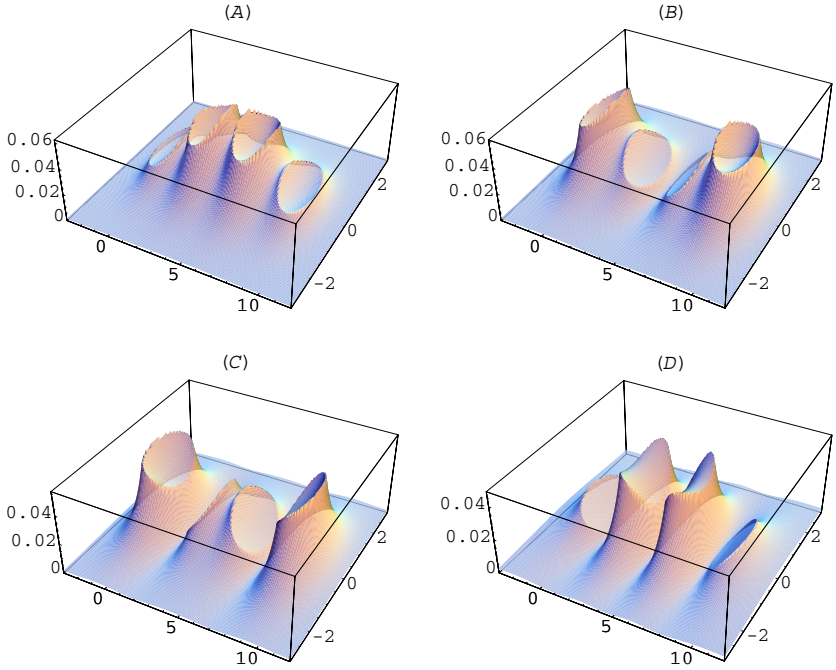

Figure 1. $m = 0$. Normalized probability density for the four states of a linear chain with four centres. A, B, C, D in order of increasing energy (see the text).

Figure 1 reports the graphs of the four normalized probability densities for λ slightly smaller than the last critical value, $\lambda = -0.66$. The eigenvalues of the Hamiltonian are $-0.364\,722$, $-0.282\,859$, $-0.154\,874$, $-0.002\,725\,88$. The first graph is for the state of lowest energy, and then successively in order of increasing energy.

4.2.2. The case $m \neq 0$ and only one sign for m. The Hamiltonian with K centres and K projectors can be written as

$$_K H^m = \frac{p^2}{2M} + \lambda \sum_{j=1}^{K} |\xi_j^m\rangle\langle\xi_j^m| \qquad |\xi_j^0\rangle = \exp(-\mathrm{i}jLp_x)r|r, m\rangle.$$

In this case also there is a one-to-one correspondence between centres and projectors. In the zero energy limit, the diagonal terms of the matrix b are given by equation (24), $1 + \frac{M\lambda r_\alpha^2}{|m_\alpha|}$. The behaviour of the non-diagonal terms is obtained from equation (28):

$$b_{\alpha\beta}(0) = -\lambda r^2 \langle r, m| \exp(\mathrm{i}ap_x) G_0(0)|r, m\rangle = 0.$$

One concludes that all the K bound states appear or disappear for the same one centre critical value $\lambda = -\frac{|m|}{Mr^2}$. This situation should be contrasted with the preceding case $m = 0$ where different bound states appear or disappear for different critical values.

Figure 2 reports the graphs of the four normalized probability densities for λ slightly smaller than the unique critical value, $\lambda = -1.1$, $M = r = 1$, $L = 3$, $m = 1$. The eigenvalues of the Hamiltonian are $-0.107\,234$, $-0.083\,3678$,$-0.054\,413$, $-0.029\,2872$. The first graph is for the state of lowest energy, and then successively in order of increasing energy.

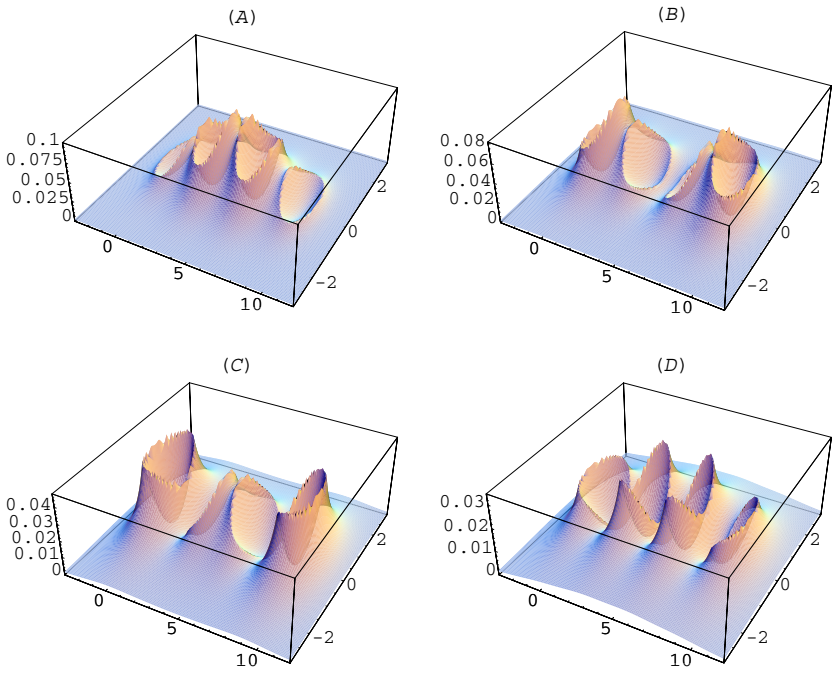

Figure 2. $m = 1$. Normalized probability density for the four states of a linear chain with four centres. A, B, C, D in order of increasing energy (see the text).

4.2.3. The case $m \neq 0$ and two signs for m. For a given centre j, both projectors with $m > 0$ and $m < 0$ are now present. Let $[u]$ denote the integer part of u, p_x the x component of the momentum and L the distance between two consecutive centres of the chain on the x axis. The Hamiltonian with K centres and $2K$ projectors can be written as

$$
\begin{aligned}
{}_{2K}H^{|m|} &= \frac{p^2}{2M} + \lambda \sum_{j=1}^{K} \left|\xi_j^m\right\rangle\left\langle\xi_j^m\right| + \left|\xi_j^{-m}\right\rangle\left\langle\xi_j^{-m}\right| \\
&= \frac{p^2}{2M} + \lambda \sum_{\alpha=1}^{2K} |\xi_\alpha\rangle\langle\xi_\alpha| \\
|\xi_\alpha\rangle &= \exp\left(-\mathrm{i}\left[\frac{\alpha+1}{2}\right] L p_x\right) r|r, (-1)^\alpha |m|\rangle.
\end{aligned}
$$

In the zero energy limit, the diagonal terms of the matrix b are given by equation (24), and for the non-diagonal terms one obtains from equations (28), (29)

$$
\begin{aligned}
b_{\alpha\beta}(0) &= -\lambda r^2 \langle r, m| \exp(\mathrm{i}ap_x) G_0(0) |r, m\rangle = 0 \\
b_{\alpha\beta}(0) &= -\lambda r^2 \langle r, m| \exp(\mathrm{i}ap_x) G_0(0) |r, -m\rangle \\
&= (-1)^m M\lambda r^2 \frac{(|2m|-1)!}{(|m|!)^2} \left(\frac{r}{a}\right)^{2|m|},
\end{aligned}
$$

where a denotes the distance between the projectors α and β.

Some algebra then shows that the equation $\det(b(0)) = 0$ reduces to the condition that

$$(-1)^{m+1}\frac{(|m|!)^2}{(|2m|-1)!}\left(\frac{L}{r}\right)^{2|m|}\left(\frac{1}{M\lambda r^2}+\frac{1}{|m|}\right)$$

be an eigenvalue of the following symmetric matrix of order $2K$:

$$\begin{array}{cccccccccc}
0 & 0 & 0 & 1 & 0 & 2^{-2|m|} & 0 & 3^{-2|m|} & \dots \\
0 & 0 & 1 & 0 & 2^{-2|m|} & 0 & 3^{-2|m|} & 0 & \dots \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 2^{-2|m|} & \dots \\
1 & 0 & 0 & 0 & 1 & 0 & 2^{-2|m|} & 0 & \dots \\
0 & 2^{-2|m|} & 0 & 1 & 0 & 0 & 0 & 1 & \dots \\
2^{-2|m|} & 0 & 1 & 0 & 0 & 0 & 1 & 0 & \dots \\
0 & 3^{-2|m|} & 0 & 2^{-2|m|} & 0 & 1 & 0 & 0 & \dots \\
3^{-2|m|} & 0 & 2^{-2|m|} & 0 & 1 & 0 & 0 & 0 & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}$$

With t_k the eigenvalues of this matrix, the critical values λ_k for which a new bound state appears or disappears are given by

$$\frac{1}{\lambda_k} = -\frac{Mr^2}{|m|}\left[(-1)^m\frac{(|2m|-1)!|m|}{(|m|!)^2}\left(\frac{r}{L}\right)^{2|m|}t_k + 1\right]. \tag{35}$$

For example, for $M = r = 1, L = 3, |m| = 1$ and a linear chain with four centres (eight projectors) the critical values are $-0.827\,037$, $-0.863\,115$, $-0.921\,083$, $-0.966\,055$, $-1.036\,42$, $-1.093\,71$, $-1.188\,49$, $-1.264\,44$.

Figure 3 reports the graphs of the eight normalized probability densities for λ slightly smaller than the last critical value, $\lambda = -1.3$. The eigenvalues of the Hamiltonian are $-0.520\,908$, $-0.379\,859$, $-0.358\,059$, $-0.303\,785$, $-0.224\,693$, $-0.144\,763$, $-0.174\,565$, $-0.004\,578\,01$. The first graph is for the state of lowest energy, and then successively in order of increasing energy from left to right.

5. Scattering

General formulation of scattering theory in two dimensions has been discussed elsewhere [16–18]. For the sake of self-consistency we briefly indicate some basic results. Since scattering theory is more familiar in the three-dimensional case, we parallel the results for three- and two-dimensional cases.

Let us begin with notations and definitions valid in both two and three dimensions. The scattering matrix is denoted by S as usual. The $T(z)$ operator is related to the interaction V and to the resolvent $G(z) = (z - H)^{-1}$ by

$$T(z) = V + VG(z)V.$$

The on-shell matrix elements of $T(z)$ are denoted by

$$t(\mathbf{p}' \leftarrow \mathbf{p}) = \lim_{\varepsilon\to 0}\langle \mathbf{p}'|T\left(\frac{p^2}{2M} + \mathrm{i}\varepsilon\right)|\mathbf{p}\rangle,$$

with $p = p'$. The relation with the S matrix elements is

$$\langle \mathbf{p}'|S|\mathbf{p}\rangle = \delta(\mathbf{p}' - \mathbf{p}) - 2\pi\mathrm{i}\delta(E_p - E_{p'})t(\mathbf{p}' \leftarrow \mathbf{p}).$$

The stationary scattering states $|\mathbf{p}+\rangle$ are defined by

$$|\mathbf{p}+\rangle = |\mathbf{p}\rangle + \lim_{\varepsilon\to 0} G\left(\frac{p^2}{2M} + \mathrm{i}\varepsilon\right)V|\mathbf{p}\rangle.$$

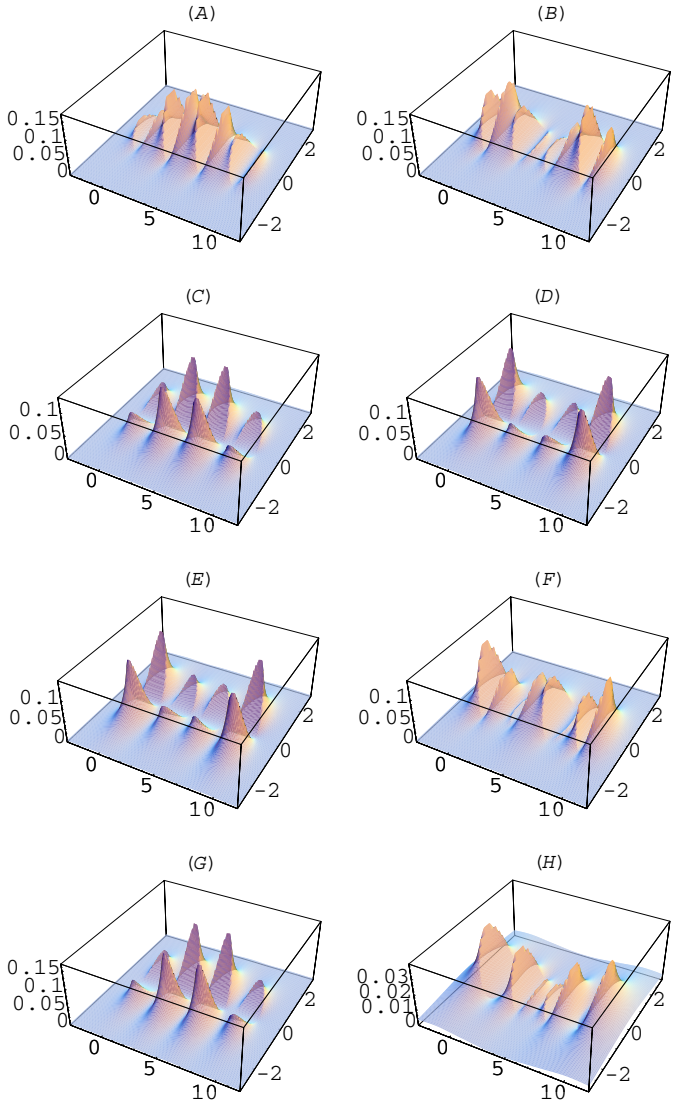

Figure 3. $m = \pm 1$. Normalized probability density for the eight states of a linear chain with four centres. A, B, C, D, E, F, G, H in order of increasing energy (see the text).

For all these results, see e.g. [19]. We now turn to results specific to each dimension. A list of formulae is now given for the three-dimensional case and the two-dimensional case. Each formula for the three-dimensional case is immediately followed by the corresponding formula for the two-dimensional case..

The scattering amplitude f is defined from the asymptotic behaviour ($r \to \infty$) of the stationary scattering wavefunctions,

$$\langle \mathbf{r}|\mathbf{p}+\rangle \sim \frac{1}{(2\pi)^{3/2}} \left\{ \exp(\mathrm{i}\mathbf{p}\cdot\mathbf{r}) + f(p\widehat{\mathbf{r}} \leftarrow \mathbf{p}) \frac{\exp(\mathrm{i}pr)}{r} \right\}$$

$$\langle \mathbf{r}|\mathbf{p}+\rangle \sim \frac{1}{2\pi} \left\{ \exp(\mathrm{i}\mathbf{p}\cdot\mathbf{r}) + f(p\widehat{\mathbf{r}} \leftarrow \mathbf{p}) \frac{\exp(\mathrm{i}pr)}{\sqrt{r}} \right\}$$

$$f(\mathbf{p}'\leftarrow\mathbf{p}) = -(2\pi)^2 M t(\mathbf{p}'\leftarrow\mathbf{p})$$
$$f(\mathbf{p}'\leftarrow\mathbf{p}) = -\exp\left(\mathrm{i}\frac{\pi}{4}\right)(2\pi)^{3/2} M\sqrt{\frac{1}{p}}\,t(\mathbf{p}'\leftarrow\mathbf{p}).$$

The analogue of the three-dimensional scattering cross section σ will be called the scattering cross length, to be denoted by l,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\mathbf{p}'\leftarrow\mathbf{p})|^2 \qquad \frac{\mathrm{d}l}{\mathrm{d}\varphi} = |f(\mathbf{p}'\leftarrow\mathbf{p})|^2. \tag{36}$$

For interaction invariant under rotation, partial wave analysis is very convenient.

$$\begin{aligned}
&f(\mathbf{p}'\leftarrow\mathbf{p}) = \sum_{\ell=0}^{\infty}(2\ell+1)f_\ell(p)P_\ell(\widehat{\mathbf{p}'}\cdot\widehat{\mathbf{p}})\\
&f(\mathbf{p}'\leftarrow\mathbf{p}) = \sum_{m=-\infty}^{\infty}\exp(\mathrm{i}m(\varphi_{\mathbf{p}'}-\varphi_{\mathbf{p}}))f_m(p)\\
&f_\ell(p) = \frac{\exp(\mathrm{i}\delta_\ell(p))\sin(\delta_\ell(p))}{p}\\
&f_m(p) = \sqrt{\frac{2}{\pi}}\exp\left(\mathrm{i}\frac{\pi}{4}\right)\sqrt{\frac{1}{p}}\exp(\mathrm{i}\delta_m(p))\sin(\delta_m(p))\\
&\sigma(p) = \int_0^{\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi|f(\mathbf{p}'\leftarrow\mathbf{p})|^2 = 4\pi\sum_{\ell=0}^{\infty}(2\ell+1)|f_\ell(p)|^2 = \sum_{\ell=0}^{\infty}\sigma_\ell(p)\\
&l(p) = \int_0^{2\pi}\mathrm{d}\phi|f(\mathbf{p}'\leftarrow\mathbf{p})|^2 = \sum_{m=-\infty}^{\infty}2\pi|f_m(p)|^2 = \sum_{m=-\infty}^{\infty}l_m(p)
\end{aligned} \tag{37}$$

$$\begin{aligned}
&\sigma_\ell(p) = 4\pi(2\ell+1)\frac{\sin^2(\delta_\ell(p))}{p^2} \leqslant \frac{4\pi(2\ell+1)}{p^2}\\
&l_m(p) = \frac{4}{p}\sin^2(\delta_m(p)) \leqslant \frac{4}{p}.
\end{aligned} \tag{38}$$

Finally, the optical theorem

$$\begin{aligned}
&\mathrm{Im}[f(\mathbf{p}\leftarrow\mathbf{p})] = \frac{p}{4\pi}\sigma(\mathbf{p})\\
&\mathrm{Re}\left[\exp\left(\mathrm{i}\frac{\pi}{4}\right)f(\mathbf{p}\leftarrow\mathbf{p})\right] = -\frac{1}{2}\sqrt{\frac{p}{2\pi}}\,l(\mathbf{p}).
\end{aligned} \tag{39}$$

It should be noted that a different definition of the scattering amplitude yields [18] a formulation of the optical theorem in two dimensions which looks like to the one in three dimensions.

5.1. Scattering by a single centre

The interaction reduces to

$$V = \sum_m \lambda^m (r^m)^2 |r^m, m\rangle\langle r^m, m|. \tag{40}$$

It has to be noted that the superscript m is an index and not a power. The T operator and partial wave scattering amplitude are

$$T(z) = \sum_m \frac{(r^m)^2 |r^m, m\rangle\langle r^m, m|}{\frac{1}{\lambda^m} - (r^m)^2 \langle r^m, m|G_0(z)|r^m, m\rangle}$$

$$f_m(p) = -\exp\left(\mathrm{i}\frac{\pi}{4}\right)\left(\frac{2}{\pi}\right)^{1/2}\sqrt{\frac{1}{p}}\frac{(J_m(pr^m))^2}{\frac{1}{\lambda^m M\pi (r^m)^2} + \mathrm{i}J_m(pr^m)H_m^{(1)}(pr^m)}.$$

Equation (37) or (39) gives

$$l(p) = \frac{4}{p}\sum_m \left|\frac{(J_m(pr^m))^2}{\frac{1}{\lambda^m M\pi (r^m)^2} + \mathrm{i}J_m(pr^m)H_m^{(1)}(pr^m)}\right|^2.$$

Equations (20)–(22) show that

$$\lim_{p\to 0} l_m(p) = \begin{cases} \infty & \text{if} \quad m = 0 \\ 0 & \text{if} \quad m \neq 0. \end{cases}$$

For $m = 0$, the result should be contrasted with the three-dimensional result where the scattering section remains finite. A divergence of the cross length at zero energy has also been found [16] for scattering by a 'hard circle'. For $m \neq 0$, the explanation in terms of classical mechanics is the same: in the limit of zero energy, the angular momentum $\mathbf{r} \wedge \mathbf{p}$ tends to zero.

Let us finally consider the limit of infinite strength parameter:

$$\lim_{\lambda^m \to \pm\infty} l_m(p) = \frac{4}{p}\left|\frac{J_m(pr^m)}{H_m^{(1)}(pr^m)}\right|^2. \tag{41}$$

Since $H_m^{(1)}(pr) = J_m(pr) + \mathrm{i}Y_m(pr)$ [12] with both $J_m(pr)$ and $Y_m(pr)$ real, it is clear that the unitary bound given in inequality (38) is satisfied. It should be noted that the right-hand side of equation (41) is exactly the cross length for scattering by a 'hard circle' [16].

Figure 4 reports the total cross length for a particle of mass unity for the two cases:

$$V_\mp = \sum_{m=-5}^{5} -(|m| \mp 0.3)|r = 1, m\rangle\langle r = 1, m| \tag{42}$$

and for the case where λ is infinite. For V_- the λ^m values $-(|m| - 0.3)$ are slightly larger than the critical values $-|m|$ (see equation (30)), and the cross length exhibits resonances. The sharpest ones at the largest energy correspond to $\pm m = 5$, and then, in order of decreasing energy, with increasing widths, $\pm m = 4, 3, 2, 1$. The broadest resonance corresponding to $\pm m = 1$ is almost invisible. It is recalled that at zero energy, the cross length diverges. For more detailed explanations and for scattering by more complex systems, the methods of [2] can be generalized.

6. The one-centre problem in a uniform magnetic field

The Hamiltonian is $\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2/2M + V$, with e the algebraic charge of the particle, c the speed of light and V the interaction given by equation (40). The magnetic field $\mathbf{B}$ is perpendicular to the plane and therefore $(\mathbf{B} \wedge \mathbf{r})^2 = B^2 r^2$. With the gauge choice

$$\mathbf{A} = \tfrac{1}{2}\mathbf{B} \wedge \mathbf{r}.$$

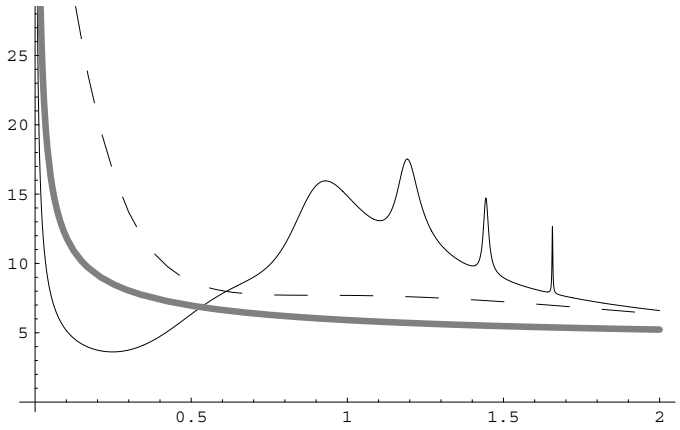

Figure 4. Ordinate: cross length l for scattering by one centre. Abscissa: the momentum of the particle. Solid curve: interaction V_-, equation (42). Dashed curve: interaction V_+, equation (42). Thick grey curve: limit $\lambda \to \pm\infty$.

$\mathbf{A}$ belongs to the plane and $\mathbf{p}\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{p} = \ell B/2$ with ℓ given by equation (1). The Hamiltonian is thus a two-dimensional one and reads [20]

$$\begin{aligned} H &= H_{0B} + V \\ H_{0B} &= \frac{p^2}{2M} - \frac{eB}{2Mc}\ell + \frac{e^2B^2}{8Mc^2}r^2 \\ &= \frac{p^2}{2M} - \frac{\omega}{2}\ell + \frac{M\omega^2}{8}r^2 \end{aligned}$$

with

$$\omega = \frac{eB}{Mc}$$

the algebraic cyclotron angular frequency. The spatial matrix elements of the resolvent

$$G_{0B}(z) \equiv \frac{1}{z - H_{0B}}$$

can be expanded in partial wave:

$$\langle \mathbf{r}_2|G_{0B}(z)|\mathbf{r}_1\rangle = \sum_{m=-\infty}^{\infty} \frac{\exp(\mathrm{i}m(\varphi_2 - \varphi_1))}{2\pi}\langle r_2, m|G_{0B}(z)|r_1, m\rangle.$$

The partial wave matrix elements will be determined from the differential equation

$$\left(z - \left\{-\frac{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}(-m^2)}{2M} - \frac{\omega}{2}m + \frac{M\omega^2}{8}r^2\right\}\right)\langle r, m|G_{0B}(z)|r_1, m\rangle = \frac{\delta(r - r_1)}{r_1},$$

with appropriate boundary conditions.

The change of variable, of function and of parameter [20],

$$\zeta = \frac{|e|B}{2c}r^2 = \frac{M|\omega|}{2}r^2 \tag{43}$$

$$f(\zeta, \zeta_1) = \langle r, m|G_{0B}(z)|r_1, m\rangle$$

$$\kappa = Mz\frac{c}{|e|B} + \frac{e}{|e|}\frac{m}{2} = \frac{z}{|\omega|} + \frac{e}{|e|}\frac{m}{2}, \tag{44}$$

yields

$$\left(\zeta\frac{\partial^2}{\partial\zeta^2}+\frac{\partial}{\partial\zeta}+\left[-\frac{m^2}{4\zeta}+\kappa-\frac{\zeta}{4}\right]\right)f(\zeta,\zeta_1)=M\delta(\zeta-\zeta_1).$$

The further change of function $g=\sqrt{\zeta}f\sqrt{\zeta_1}$ yields

$$\left\{\frac{\partial^2 g}{\partial\zeta^2}+\left[-\frac{1}{4}+\frac{\kappa}{\zeta}+\frac{\frac{1}{4}-\frac{m^2}{4}}{\zeta^2}\right]g\right\}=M\delta(\zeta-\zeta_1). \tag{45}$$

This equation is a particular case of the Sturm–Liouville problem, and a solution for g can be expressed [14, 15] in terms of two linearly independent solutions g_1, g_2 of the homogeneous equation, and of their Wronskian $\mathcal{W}(g_1,g_2)=g_1g_2'-g_1'g_2$:

$$g=M\frac{g_1(\zeta_<)g_2(\zeta_>)}{\mathcal{W}(g_1,g_2)}.$$

The homogeneous equation ($\zeta\neq\zeta_1$) is Whittaker's differential equation. A solution regular at origin is $M_{\kappa,\frac{|m|}{2}}(\zeta)$, and a linearly independent solution decreasing exponentially at infinity is $W_{\kappa,\frac{|m|}{2}}(\zeta)$. The Wronskian $\mathcal{W}$ is [21]

$$W\left(W_{\kappa,\frac{\mu}{2}},M_{\kappa,\frac{\mu}{2}}\right)=\frac{\Gamma(1+\mu)}{\Gamma\left(\frac{1+\mu}{2}-\kappa\right)}.$$

The final result then is

$$\langle r,m|G_{0B}(z)|r_1,m_1\rangle=-\delta_{mm_1}M\frac{\Gamma\left(\frac{1+|m|}{2}-\kappa\right)}{\Gamma(1+|m|)}\frac{M_{\kappa,\frac{|m|}{2}}(\zeta_<)}{\sqrt{\zeta_<}}\frac{W_{\kappa,\frac{|m|}{2}}(\zeta_>)}{\sqrt{\zeta_>}}, \tag{46}$$

with ζ given by equation (43) and κ by equation (44).

At this point it may be of interest to recall that a beautiful analytic result for the full Green function $\langle\mathbf{r}|G_{0B}(z)|\mathbf{r}_1\rangle$ has been obtained by Dodonov *et al* [22]. Equation (46) represents a Fourier component of $\langle\mathbf{r}|G_{0B}(z)|\mathbf{r}_1\rangle$. Explicitly,

$$\begin{aligned}\langle r,m|G_{0B}(z)|r_1,m\rangle&=\int_0^{2\pi}\langle\mathbf{r}|G_{0B}(z)|\mathbf{r}_1\rangle\exp(-\mathrm{i}m(\varphi-\varphi_1))\,\mathrm{d}(\varphi-\varphi_1)\\&=-\frac{M}{2\pi}\Gamma\left(\frac{1}{2}-\frac{z}{|\omega|}\right)\int_0^{2\pi}\exp\left[\mathrm{i}\left(\frac{M\omega}{2}rr_1\sin(\varphi)-m\varphi\right)\right]\\&\frac{W_{\frac{z}{|\omega|},0}\left(\frac{M|\omega|}{2}\left(r^2+r_1^2-2rr_1\cos(\varphi)\right)\right)}{\sqrt{\frac{M|\omega|}{2}\left(r^2+r_1^2-2rr_1\cos(\varphi)\right)}}\,\mathrm{d}\varphi.\end{aligned}$$

Figure 5 reports the energies as functions of the magnetic field intensity B (measured in units of $|\omega|$) for the following Hamiltonian:

$$H=H_{0B}+\sum_{m=-5}^{5}\lambda^{|m|}|r=1,m\rangle\langle r=1,m|,$$

with $M=1$, $e=-1$ and $\lambda^{|m|}$ defined by the numerical values given in equation (31). It is recalled that these values have been computed in order that all eigenvalues of the Hamiltonian are degenerated at zero field where they take the value -1. For each m value, the energy E is determined as the root of

$$\frac{1}{\lambda^{|m|}}=r^2\langle r,m|G_{0B}(E)|r,m\rangle.$$

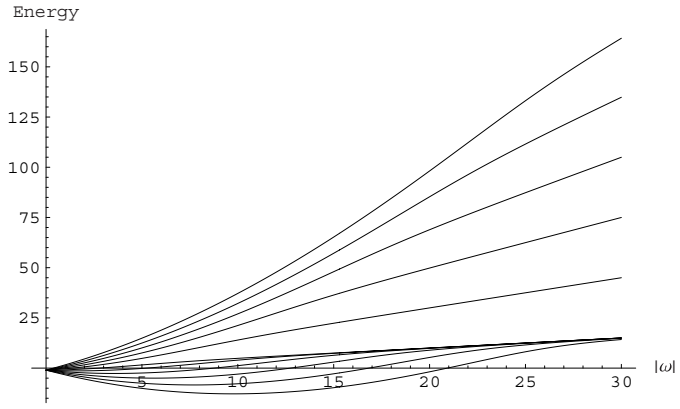

Figure 5. Energies as a function of the magnetic field intensity in unit of $|\omega|$ (see the text).

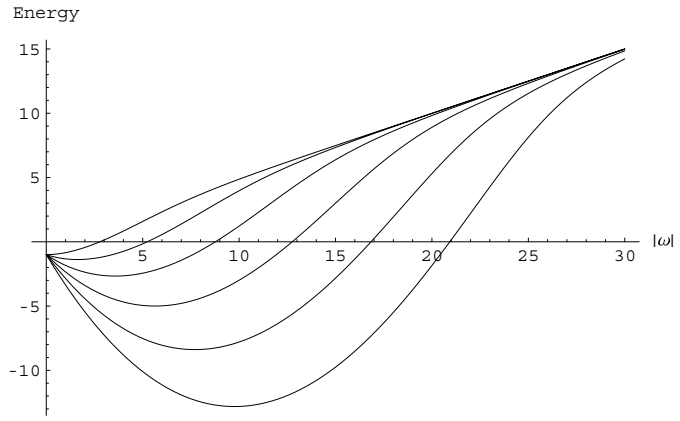

Figure 6. Enlarged view of figure 5 for $m \leqslant 0$ (see the text).

One clearly sees a linear behaviour for small field, asymptotics to linear atomic Zeeman effect, $E = -1 + m|\omega|/2$. At the other limit $B \to \infty$, the 11 energy levels are asymptotic to the 6 Landau levels $|m + 1/2||\omega|$. Specifically, the highest level, $m = 5$, is asymptotic to $(11/2)|\omega|$, the $m = 4$ level to $(9/2)|\omega|$, the $m = 3$ level to $(7/2)|\omega|$, the $m = 2$ level to $(5/2)|\omega|$, and the $m = 1$ level to $(3/2)|\omega|$. The six other levels, from $m = 0$ to $m = -5$ in order of decreasing energy, all are asymptotic to the ground state Landau level, $|\omega|/2$. An enlarged view for these six levels corresponding to $m \leqslant 0$ is reported in figure 6 for a better visualization.

7. Discussion and conclusion

The present model can be applied to study the spectrum of finite two-dimensional complex systems involving arbitrary geometrical configurations: truncated periodic, truncated quasi-periodic, disordered. It can also be applied to the study of collisional processes involving two-dimensional targets with complex geometrical configurations, or for analysing wave packets time evolution on plane nano-structures. The flexibility of this model relies also on

the possibility of choosing not only the geometrical configuration of the system, but also the parameters of each centred interactions. The strength parameters λ_j^k, range parameters r_j^k, angular momentum parameters m_j^k can be varied at will (with the restriction of non-overlapping interactions) and in an independent way.

For very large systems, the numerical search of zeros which is necessary for the determination of density of states may become difficult due to numerical precision effects. In that case, qualitative information on the statistic of the energy levels distribution could be obtained by solving the eigenvalue problem not for the energy, but for the strength parameters λ_j^k with zero energy as boundary condition. The practical solution of this problem indeed reduces to the numerical determination of eigenvalues of Hermitic large matrices with very simple matrix elements, a task much easier than the search of zeros of determinants of Hermitic large matrices with matrix elements involving Bessel functions.

The present model could also be used to study of time-dependent phenomena. The determination of energy surfaces as a function of the geometrical parameters without too much numerical efforts should indeed be of interest for model studies of geometrical phase effects (Berry's adiabatic phase and Aharonov–Anandan phase). In this context the inclusion of a time-dependent uniform magnetic field would be of special interest. At present however, in the presence of non-zero uniform magnetic fields, an analytic expression for the matrice elements of the resolvent has been obtained only for matrices elements involving the same centre (equation (46)).

Finally, it has to be noted that two-dimensional physics is relevant not only for surface physics but also for other systems such as electrons in semiconductor heterojunctions which can exhibit two-dimensional or quasi-two-dimensional behaviour [23].

Appendix. Evaluation of a class of integrals

We shall prove the following property.

(i) If n and f have the same parity (i.e. n even number and f even function or n odd number and f odd function),
(ii) if f is holomorphic in the upper half plane, and
(iii) if, in the limit of infinite radius R, the integral along a semicircle D centred at the origin and located in the upper half plane, $\int_D \mathrm{d}z\, \frac{zH_n^{(1)}(az)f(z)}{z^2-z_0^2} \to 0$,

then for $a > 0$, and z_0 the squared root of z_0^2 with positive imaginary part:

$$\int_0^\infty \mathrm{d}z\, \frac{zJ_n(az)f(z)}{z^2-z_0^2} = \frac{\pi \mathrm{i}}{2} H_n^{(1)}(az_0)f(z_0).$$

As the integrand is odd, the procedure used in [2] for which even integrand g yields $\int_0^\infty \mathrm{d}z\, g(z) = \frac{1}{2}\int_{-\infty}^\infty \mathrm{d}z\, g(z)$ cannot be applied. To proceed further, let us consider the following integral:

$$A_R = \int_{C_R} \mathrm{d}z\, \frac{zH_n^{(1)}(az)f(z)}{z^2-z_0^2},$$

where C_R is the closed contour from $-R$ to R and the upper semicircle of radius R. In the interval $[-R, 0]$, the value of $H_n^{(1)}$ is the one obtained by continuity from the upper plane above the cut $]-\infty, 0[$ pertaining to the function $H_n^{(1)}$. Then, within the hypotheses of the property, the residue theorem yields

$$\lim_{R\to\infty} A_R = 2\pi \mathrm{i} \frac{z_0 H_n^{(1)}(az_0)f(z_0)}{2z_0} = \int_{-\infty}^\infty \mathrm{d}z \frac{zH_n^{(1)}(az)f(z)}{z^2-z_0^2}.$$

Splitting the integral into $\int_{-\infty}^{0} + \cdots + \int_{0}^{\infty}$, changing z into $-z$ in the integral from $-\infty$ to 0:

$$\pi \mathrm{i} \frac{z_0 H_n^{(1)}(az_0) f(z_0)}{z_0} = -\int_0^\infty \mathrm{d}z \, \frac{z H_n^{(1)}(-az) f(-z)}{z^2 - z_0^2} + \int_0^\infty \mathrm{d}z \, \frac{z H_n^{(1)}(az) f(z)}{z^2 - z_0^2}.$$

Noting [12] that for $az > 0$,

$$H_n^{(1)}(-az) = (-1)^n \left[H_n^{(1)}(az) - 2J_n(az) \right]$$

one obtains taking into account that f and n have the same parity

$$\pi \mathrm{i} \frac{z_0 H_n^{(1)}(az_0) f(z_0)}{z_0} = 2 \int_0^\infty \mathrm{d}z \frac{z J_n(az) f(z)}{z^2 - z_0^2},$$

which gives the final result.

Equations (16) and (17) then follow because

(i) the relation

$$J_m(-z) = (-1)^m J_m(z)$$

ensures that condition 1 of the property is satisfied.

(ii) J_m is holomorphic and therefore condition 2 of the property is satisfied.

(iii) The asymptotic behaviours

$$J_m(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4} \right) \tag{A.1}$$

$$H_m^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} \exp\left[\mathrm{i} \left(z - \frac{m\pi}{2} - \frac{\pi}{4} \right) \right] \tag{A.2}$$

ensure that condition 3 of the property is satisfied.

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